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SPHEICAL GEOMETRY.

By EDWIN BIDWELL WILSON, Yale University.

LECTURE II. THE ELEMENTS OF THE THEORY OF CONGRUENCE AND OF CONTINUITY.

Thera are a number of theorems on congruence and on continuity which are as self-evidident as the axioms on which they depend. The difficulty is not in giving the proof but in seeing wherein lies the necessity or possibility of proof. It must be remembered, however, that the difference between an axiom and a theorem is not that one is more readily granted as self-evident than the other. Often a theorem may be just as evidently true as is any axiom. The difference is this: An axiom is a more or less self-evident proposition arbitrarily selected selected from among a large number of such propositions to serve as a basis for future logical reasoning, whereas a theorem is a proposition which may be deduced by a number of logical steps or syllogisms from the assumed axioms. The axioms are merely a set of general statements or major premises which are necessary in order to start the chain of reasoning of which the theorems are the final results. From this it appears that a theorem possesses truth or validity only in reference to the axioms assumed and hence it is important that the axioms be as well founded as possible and in particular be not contradictory with one another. It is further clear that the rôles of the elementary theorems and of the axioms may be interchanged to a large extent, theorems becoming axioms and axioms then becoming theorems.

Theorem 1. *If in a motion one point and a direction issuing from it remain fixed all points remain fixed.*

To state this in symbolic form: If $A \equiv A$ and $a \equiv a$ (where it is understood that the small letter indicates a half-line or direction issuing from the point represented by the corresponding large letter), then $P \equiv P$ for every point P on the spherical surface. To prove this we note that it has been stated in Lecture I that the simplest of all motions is the identical transformation in which each point keeps its original position. In this particular motion, then, $A \equiv A$ and $a \equiv a$ and $P \equiv P$. Axiom V_3 states that the correspondence of a point and direction to a point and direction completely and uniquely determines the motion. Hence any motion in which A, $a \equiv A$, a can be no other than the identical transformation and $P \equiv P$. The theorem is therefore proved.

Theorem 2. *To every rigid motion there exists an inverse rigid motion.*

The inverse of any transformation produces just the opposite effect of the transformation itself. If the points of the surface have been moved by a certain motion from one position to a second position, the inverse motion moves them all back to the original position. The theorem states that no matter how the points have been moved about, it is always possible to find a motion which will take them back to their first positions. Let $\mathbf{M}$ stand for a rigid motion. Let A be any point and a any direction issuing from it. Suppose that the motion $\mathbf{M}$ carries A into A' and a into a', then by Axiom V_3 the motion is thereby determined.[*] By this same axiom we are assured that we may inversely carry A' into A and a' into a by some motion $\mathbf{M}'$. From Axiom V_1 it follows that the result of two successive motions such as $\mathbf{M}$ and $\mathbf{M}'$ produce another motion $\mathbf{M}''$. This, however, is the identical transformation; for A and a have been carried into A' and a', and then back into A and a, so that Theorem 1 applies. Hence by the succession of $\mathbf{M}$ and $\mathbf{M}'$ every point P of the surface has been returned to its original position and $\mathbf{M}'$ is the inverse of $\mathbf{M}$. The theorem is therefore proved.

This theorem is instructive in a number of ways. To begin with it is fully as evident as any axiom and might have been included among them. The proof tends to render the theorem obscure rather than more clear and unless given with extraordinary care is almost certain to be not valid,—to assume tacitly more than the axioms state and more than is equivalent to the outright assumption of the theorem itself. And it is in precisely this point that the chief value of such proofs lies. Until one has given proofs for these seemingly axiomatic theorems one does not come to a full realization of what a proof is nor to a thorough understanding of just what the axioms do contain. The investigators into the foundations of geometry strive to be rid of all save the smallest possible number of axioms, to give proofs for all other propositions. This sometimes leads to great complication: for, the fewer our premises, the more elaborate must be our demonstration. From the pedagogic standpoint it is undoubtedly necessary to avoid the most difficult of these proofs and thus to posit more axioms than are of a strictly logical necessity.

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[*] This step relies on Axiom V_2. We tacitly assume that any motion will carry a point and a direction issuing from it into a point and a direction issuing therefrom. That a point is carried into a point follows from V_2. That a line passing through a point is carried over into a line passing through the corresponding point may easily be proved as follows A given line is carried over into a line (Axiom V_2) and hence the points upon the given line must be carried over into points upon the resulting line. The statement in the text, however, requires more than this. It requires that a *half*-line shall be carried over into a *half*-line, not merely that a line shall be carried over into a line. Conceivably a line might be carried into a line in such a manner that the points, while remaining on the line, changed their relative positions. Thus the points which were originally on a certain side of a point A might, after the motion, be distributed upon both sides of A. In such a case we could not speak of a half-line being carried into a half-line, for it would be split up and put part on one side and part on the other side of the point. To set aside this possibility we may change Axiom V_2 so as to read "and half-lines, into half-lines" or better, we may add, *we shall understand the term line to include the fundamental properties of order and continuity as set forth in Axioms III and IV.* Thus Axiom V_2 states that lines are carried over into lines without destroying the properties, order and continuity.

For us the above theorem serves also as an excellent introduction to the theory of transformations and to the notations therein employed. A transformation, in this case a rigid motion, may conveniently be denoted by a letter placed in brackets. The expression "the point A and the direction a are carried by the motion $\mathbf{M}$ into A' and a' which in turn are carried back by $\mathbf{M}'$ into A and a" would be written in symbols as

$$A;\ a\,[\mathbf{M}]\,A';\ a'\,[\mathbf{M}']\,A;\ a.$$

The intermediate set of letters may be omitted and the expression becomes

$$A;\ a\,[\mathbf{M}]\,[\mathbf{M}']\,A;\ a \quad \text{or } A;\ a\,[\mathbf{M}\,\mathbf{M}']\,A;\ a,$$

which is read "A and a are carried by the succession of $\mathbf{M}$ and $\mathbf{M}'$ into A and a. The inverse motion is denoted as a reciprocal, and the identical transformation is represented by unity,

$$\mathbf{M}', \text{ the inverse of } \mathbf{M}, \text{ is } \mathbf{M}^{-1};\ \mathbf{M}\mathbf{M}^{-1}=1.$$

Theorem 3. *The whole and the part of a segment cannot be congruent.*

Special Case. Let AB be any segment and B' any point of it, to show that

$$AB \text{ not} \equiv AB'.$$

Consider the direction $AB'B$. This remains fixed. The point A also remains fixed. By Theorem 1 all points must remain fixed. Hence B' cannot be carried into B. In a similar manner it can be shown that, if B' lies in AB produced,

$$AB \text{ not} \equiv AB'.$$

General Case. Let AB be any segment and $A'B'$ any segment which includes AB or is included by it. To show that the two cannot be congruent use may be made of the principles of order and of continuity. The proof will doubtless appear somewhat unfamiliar to those acquainted with only the ordinary presentations of geometry. It is nevertheless no more difficult than the proof of the theorems on limits which are included in all treatments of elementary geometry—that is, if those proofs are correctly given instead of wrongly as is usually the case. By Theorem 2, $A'B'$ must be congruent to AB if AB is congruent to

$A'B'$. Consequently it may be assumed without loss of generality that $A'B'$ is included by AB. Suppose **M** is the transformation which carries AB into $A'B'$. Consider the points $A''B''$ into which $A'B'$ are carried, the points into which these points $A''B''$ are carried, and so on. We have

$$AB\,[\mathbf{M}]\,A'B',\quad A'B'\,[\mathbf{M}]\,A''B'',\quad A''B''\,[\mathbf{M}]\,A'''B''',\ldots\ldots$$

or

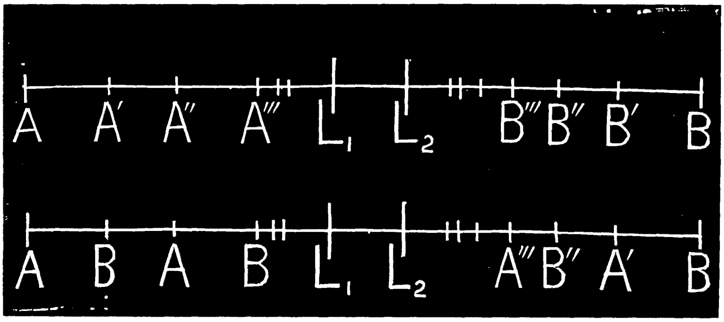
$$AB\,A'B'\,A''B''\ldots\ldots\,[\mathbf{M}]\,A'B'\,A''B''\,A'''B'''\ldots\ldots$$

According to Axiom V_2, as explained in the footnote to Theorem 2, lines are transformed into lines in such a manner as to preserve the property of order. Hence

$$AA'B,\quad A'A''B',\quad A''A'''B'',\ldots\ldots$$

have the same order upon the line AB. The order opposite to this is shared in common by the series of points

$$BB'A,\quad B'B''A',\quad B''B'''A'',\ldots$$



As shown in Fig. 1, two different cases may occur. In the first of them the points

$$A,\ A',\ A'',\ A''',\ldots\ldots$$

move constantly in the same direction and the points

$$B,\ B',\ B'',\ B''',\ldots\ldots$$

in the opposite direction. In the second case the points

$$A,\ B',\ A'',\ B''',\ldots\ldots$$

move in one direction and the points

$$B,\ A',\ B'',\ A''',\ldots\ldots$$

in the direction opposite to them. In either case on examining the order of $AA'B'$, $A'A''B''$, $A''A'''B''$, etc., it is seen that the points of the first set lie before these of the second set. We have, then, two series of points continually approaching but never passing one another. Under these circumstances it is geometrically evident (and it will be proved in the next theorem) that the points of each set must approach a limiting point which may be the same for both sets. Let L_1 be the limit of the first set in the first case. L_1 is not changed in position by the motion **M**. On the proof of this turns the demonstration of the theorem. L_1 cannot move to the left; for if it did it would fall in one of the segments $A^{(n)}A^{(n+1)}$ and we know that the points which make up this segment have come from the segment $A^{(n-1)}A^{(n)}$. L_1 cannot move to the right; for if it did any point X situated between the original and the new positions of L_1 would have to come from some point to the left of L, that is, from some segment $A^{(n)}A^{(n+1)}$ all of whose points we know go into the segment $A^{(n+1)}A^{(n+2)}$. Hence in the first case L_1 remains at rest. Therefore L_1A [**M**] L_1A', which is contradictory to what has been shown in the special case of this theorem. As we have deduced by logical steps an impossibility from reasoning on the assumption that AB may be carried into $A'B'$, it is evident that that assumption must have been wrong. To treat the second case it is best not to use the transformation **M**, but **M** followed by **M**. Then the points with an odd accent may be neglected.

$$ABA''B''\ldots\ldots[\mathbf{M}]\,[\mathbf{M}]\,A''B''A''''B''''\ldots\ldots\ldots\ldots$$

For the transformation **M M** this case offers the same difficulty as the former did for **M**. Therefore the second supposition is also impossible and the theorem that a whole and a part of a segment cannot be congruent is proved in its entirety.

Theorem 4. *Given a series of collinear points A, A', A'', ... the members of which move continually in the same direction without, however, passing a certain point B. There must exist a limiting point L to this series and this limiting point does not lie beyond B.**

Consider any point X of the segment AB. Either there are members of the series between X and B or there are not. Separate the points of the segment into two classes corresponding to these two assumptions. Let X be any point of the first class and X' any point of the second. There is some point $A^{(n)}$ of the series between X and B. Hence X comes before $A^{(n)}$; but X' cannot come before $A^{(n)}$. Hence X comes before X' and every point of the first class precedes every point of the second class. By Axiom IV, there exists a point L such that any point before L belongs to the first class and any point after L belongs to the second. To show that L is the limiting point of the series it is necessary to have some perfectly definite definition of limiting point. The idea of a limit is generally connected with that of nearness. The statement of the definition is expressed in terms of distance. For us at the present stage of development this is impossible because no measure of distance has been adopted and indeed it has not yet been stated how distance can be measured. A more general definition of the limiting point, one that is independent of distance, must be given. If L be the limiting point of a series of points situated on a line, and if we choose from the line any segment which contains in its interior the point L, then from a certain point on all the points of the series will lie in this segment. Conversely, *if, when an arbitrary segment surrounding the point L has been chosen, all the points of the series from some point on lie in this segment, then the point L is said to be the limiting point of the series.* In case the points of the series all lie on the same side of the point L the segment does not surround L but may terminate in L and lie on that side of L on which the series lies. Furthermore,

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*Figure 1, or one quite similar to it, may serve the purpose.

when the points of the series always move in the same direction on the line it is sufficient to show that one point of the series lies in the arbitrarily chosen segment: for then all the rest must lie in it. It is this last, simplest state of affairs which obtains in the case in hand. Let L be the point which separates the class of points X from the class of points X' as stated above. Let LM be any segment lying on that side of the point L on which the points $A, A', A'', \ldots\ldots$ lie. The point M belongs to the first class and hence there must be some point $A^{(n)}$ of the series between M and B. On the other hand there are no such points between L and B. Hence the point $A^{(n)}$ lies in the segment LM. By the definition given above the point L is therefore the limiting point of the series $A, A', A'', \ldots\ldots$ and the theorem is proved.

The proofs of Theorems 3 and 4 required the use of the principle of continuity. When the demonstrations are critically examined it will be seen that three essential ideas are involved: first, the separation of the points of a segment into two classes such that any point of one lies bofore any point of the other; second, the application of the axiom of continuity to establish the existence of a certain point L; third, the proof that this point has the desired properties. This sort of demonstration occurs so frequently in any careful development of geometry that it is well for the reader to become thoroughly familiar with it at an early stage. A little practice will render it both clear and easy. There is no necessity of tacitly assuming that a circle is a continuous curve—the fact may be proved. In like manner it may be proved that a segment may be divided into any desired number of parts. In Euclidean geometry such a proof is scarcely necessary, for a construction may be found to accomplish that division. In spherical geometry, however, no such construction can be found and recourse must be had to the principle of continuity to show that such a thing as the third part of a segment exists.

Theorem 5. *A segment is congruent to itself reversed in direction and in this congruence one point of the segment remains fixed so that the portions of the segment on the opposite sides of the point merely change places.*

Let AB be a segment. To prove the first part of the theorem it is only necessary to note that the point A may be applied to the point B and the direction AB to the direction BA (Axiom V_3). Then B will fall upon A: for otherwise the part and the whole would be congruent. To prove the second statement of the theorem the positions of corresponding points of the segments AB and BA may be considered. The details of this demonstration are left to the reader. The entire theorem may be proved at once by having recourse to the principle of continuity. Let O be any point of AB. If O be held fixed and the direction OB brought into coincidence with the direction OA, the point B' into which the point B is carried lies either between A and B or at A or beyond A. Divide the points of the segment AB into two classes such that to the first class belong those points O for which B' lies between A and B. By the application of Axiom IV we may show that there exists a point L which remains fixed and which has the property that $LA \equiv LB$. The details of the proof are left to the reader.